Sydney Grammar School
Mathematics Department
Half-Yearly Examinations 2005

FORM VI

MATHEMATICS EXTENSION 1

Examination date

Wednesday 18th May 2005

Time allowed

2 hours

Instructions

All seven questions may be attempted.
All seven questions are of equal value.
All necessary working must be shown.
Marks may not be awarded for careless or badly arranged work.
Approved calculators and templates may be used.
A list of standard integrals is provided at the end of the examination paper.

Collection

Write your candidate number clearly on each booklet.
Hand in the seven questions in a single well-ordered pile.
Hand in a booklet for each question, even if it has not been attempted.
If you use a second booklet for a question, place it inside the first.
Keep the printed examination paper and bring it to your next Mathematics lesson.

Checklist

SGS booklets: 7 per boy. A total of 1000 booklets should be sufficient.
Candidature: 122 boys.

Examiner

KWM

QUESTION ONE (12 marks) Use a separate writing booklet. **Marks**

(a) When the polynomial $P(x) = kx^3 + x^2 - (2k-1)x + 2$ is divided by $(x+1)$, the remainder is 4. Find the value of k. **1**

(b) Differentiate the following with respect to x.

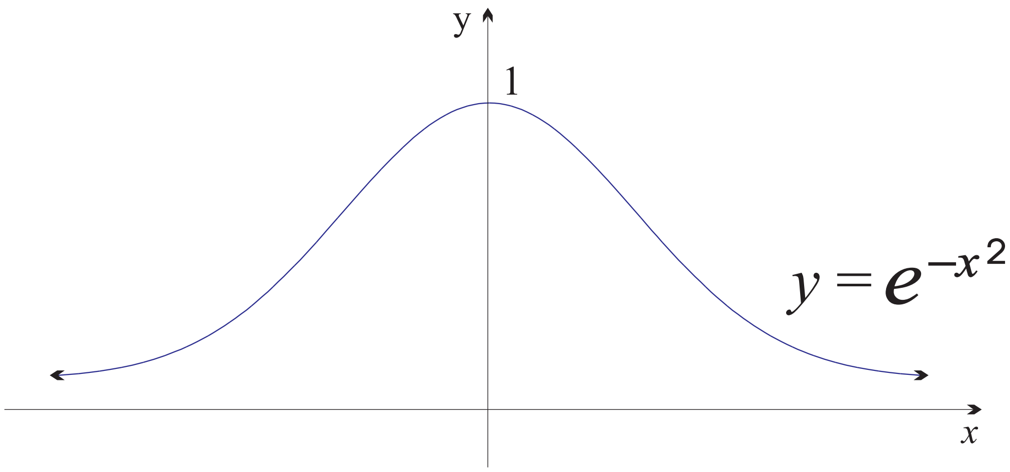
(i) $y = e^x \log_e x$ **1**

(ii) $y = \sin^{-1} 2x$ **2**

(c) Given that $\tan\alpha = \frac{1}{4}$ and $\tan\beta = \frac{3}{5}$, find the exact value of $\tan(\alpha+\beta)$. **2**

(d) Given that $x(2x-1)(x+1)+3 = 2x^3 + bx^2 + cx + 3$, for all values of x, find the values of b and c. **2**

(e)



The diagram above shows the curve $y = e^{-x^2}$.

(i) Find $\dfrac{dy}{dx}$. **1**

(ii) Show that $\dfrac{d^2y}{dx^2} = 2e^{-x^2}(2x^2-1)$. **1**

(iii) For what values of x is the curve concave down? **2**

Exam continues next page ...

QUESTION TWO (12 marks) Use a separate writing booklet. **Marks**

(a) The displacement x metres of a particle moving in a straight line is given by

$$x = 5 + 4t - t^2$$

where t is the time in seconds.

(i) Where is the particle initially? **1**

(ii) Where does the particle change direction? **2**

(iii) Find the distance travelled by the particle during the first 6 seconds. **1**

(b) (i) Without using calculus, sketch the polynomial function $y = x^3(x-1)^2$. **2**

(ii) Solve the inequation $x^3(x-1)^2 > 0$. **1**

(c) Calculate the area of the region bounded by the curve $y = \dfrac{1}{x^2+1}$ and the x-axis, between $x = 0$ and $x = \sqrt{3}$. **2**

(d) Find the following indefinite integrals.

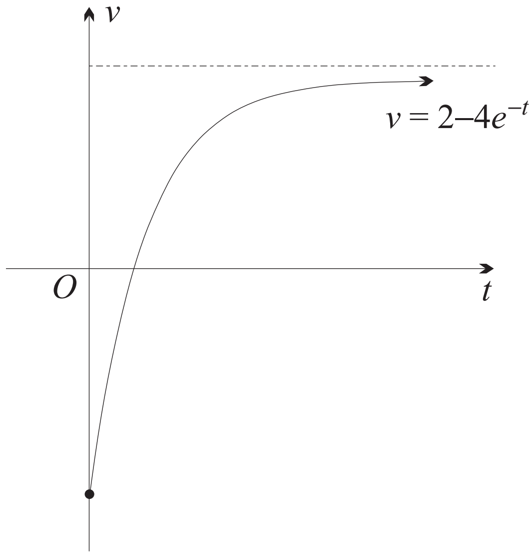
(i) $\displaystyle\int \sec^2 3x \, dx$ **1**

(ii) $\displaystyle\int \sin^3 x \cos x \, dx$ **2**

QUESTION THREE (12 marks) Use a separate writing booklet. **Marks**

(a) Solve $\cos^2 x - \sin^2 x = \dfrac{\sqrt{3}}{2}$, for $0 \leq x \leq 2\pi$. **3**

(b)



The graph above shows the velocity $v = 2 - 4e^{-t}$ m/s at time t seconds of a particle moving in a straight line.

(i) Find the initial velocity of the particle. **1**

(ii) Show that the particle comes to rest at $t = \ln 2$ seconds. **1**

(iii) Calculate the exact distance travelled by the particle before it comes to rest. **2**

(iv) To what value does the velocity of the particle approach as time increases? **1**

(c) (i) Given that $f(x) = \ln\left(\dfrac{1 + \sin x}{\cos x}\right)$, use the laws of logarithms to show that **2**

$$f'(x) = \sec x.$$

(ii) Hence or otherwise find $\displaystyle\int_0^{\frac{\pi}{4}} \sec x \, dx$. **2**

QUESTION FOUR (12 marks) Use a separate writing booklet. **Marks**

(a) Given that α, β and γ are the roots of the equation $x^3 - 4x^2 + 3x - 1 = 0$, find the value of :

(i) $\alpha + \beta + \gamma$ $\boxed{1}$

(ii) $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma}$ $\boxed{1}$

(iii) $\alpha^2 + \beta^2 + \gamma^2$ $\boxed{2}$

(b) A particle is moving in simple harmonic motion about the origin on the x-axis. Its displacement in centimetres from the origin at any time t seconds is given by

$$x = 6\cos 2t.$$

(i) Calculate the maximum velocity of the particle. $\boxed{1}$

(ii) Find the first time the acceleration of the particle is zero. $\boxed{2}$

(iii) Express the acceleration as a function of the displacement. $\boxed{1}$

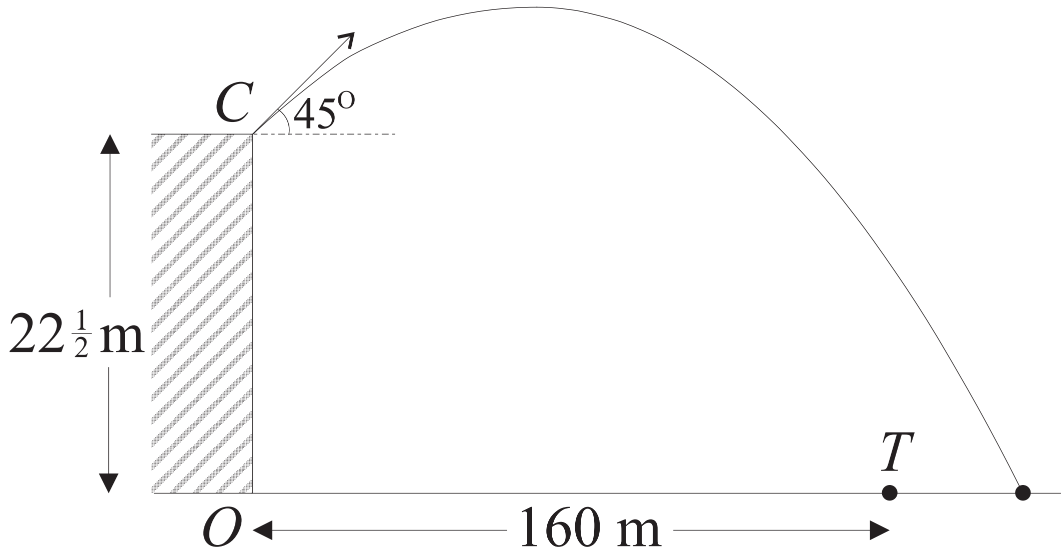
(c) Consider the curve $y = \sin^{-1} x$.

(i) Sketch the curve showing all relevant features. $\boxed{2}$

(ii) Express x as a function of y, giving any restrictions on y. Hence calculate the area of the region bounded by the curve, the x-axis and the line $x = 1$. $\boxed{2}$

Exam continues overleaf ...

QUESTION FIVE (12 marks) Use a separate writing booklet. **Marks**



The diagram above shows a cannon C positioned on the edge of a vertical cliff of height $22\frac{1}{2}$ metres. The cannon fires a shell with a muzzle velocity of 40 m/s. Set the origin O at the base of the cliff and use $g = 10$ m/s^2.

(a) Suppose that the cannon fires a shell at an angle of elevation of 45°.

(i) Show that the shell's trajectory is defined by the equations **3**

$$x = 20t\sqrt{2}$$

and $$y = 20t\sqrt{2} - 5t^2 + 22\tfrac{1}{2}.$$

(ii) Show that the shell hits the ground 180 metres from the foot of the cliff. **2**

(iii) Calculate the maximum height above the ground reached by the shell. **2**

(iv) Given that θ is the acute angle at which the shell strikes the ground, show that $\tan\theta = \frac{5}{4}$. **2**

(b) Suppose now that the angle of elevation is altered. Calculate the smaller angle of projection α, correct to the nearest minute, required for the shell to hit a target T positioned on the ground 160 metres from the base of the cliff. You may use the equations **3**

$$x = 40t\cos\alpha$$

and $$y = 40t\sin\alpha - 5t^2 + 22\tfrac{1}{2}.$$

Exam continues next page ...

QUESTION SIX (12 marks) Use a separate writing booklet. **Marks**

(a) The velocity v of a particle moving in a straight line is given by

$$v^2 = 6 + 10x - 4x^2,$$

where x is the displacement.

(i) Find the acceleration as a function of the displacement, and hence show that the motion is simple harmonic. **1**

(ii) (α) Find the centre of motion. **1**

(β) Find the period of the motion. **1**

(γ) Find the amplitude of the motion. **2**

(b) Given that $\displaystyle\int_0^k \frac{6}{\sqrt{25 - 9x^2}}\,dx = \frac{\pi}{3}$, find the value of k. **3**

(c) Use the substitution $t = \tan\frac{\theta}{2}$ to find the general solution of the equation **4**

$$\cos\theta + \tan\tfrac{\theta}{2} - 1 = 0.$$

QUESTION SEVEN (12 marks) Use a separate writing booklet. **Marks**

(a) When a polynomial $P(x)$ is divided by $(x - 1)$, the remainder is 3. When $P(x)$ is divided by $(x + 2)$, the remainder is -2. Find the remainder when the polynomial is divided by $x^2 + x - 2$. **2**

(b) The tangent to the curve $y = x^3 - 4x^2 - x + 2$, at a point Q on the curve, intersects the curve again at $A(2, -8)$. Find the co-ordinates of the point Q. **3**

(c) Rationalise the numerator of the expression $\sqrt{\dfrac{1+x}{1-x}}$ and hence find the indefinite integral $\displaystyle\int \sqrt{\frac{1+x}{1-x}}\,dx$. **3**

(d) (i) Use the definition of the derivative $f'(x) = \displaystyle\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ to find the derivative of $f(x) = e^x$ at $x = 0$, and hence show that $\displaystyle\lim_{h\to 0}\frac{e^h - 1}{h} = 1$. **2**

(ii) Find $\displaystyle\lim_{n\to\infty}\frac{e^{\frac{1}{n}} + e^{\frac{2}{n}} + e^{\frac{3}{n}} + \cdots + e^{\frac{n}{n}}}{n}$. **2**

END OF EXAMINATION

FORM VI 3 UNIT SOLUTIONS. HALF YEARLY 2005

QUESTION 1

(a) $P(x) = Kx^3 + x^2 - (2K-1)x + 2$

$P(-1) = 4$ (Using the remainder theorem.)

$-K + 1 + (2K-1) + 2 = 4$

$K + 2 = 4$ ✓

$K = 2$

(b)

(i) $y = e^x \ln x$

$\frac{dy}{dx} = e^x \ln x + e^x \times \frac{1}{x}$ ✓

$= e^x\left(\ln x + \frac{1}{x}\right)$

(ii) $y = \sin^{-1} 2x$

$\frac{dy}{dx} = \frac{1}{\sqrt{1-4x^2}} \times 2$ ✓ ✓

$= \frac{2}{\sqrt{1-4x^2}}$

(c) $\tan\alpha = \frac{1}{4}$ and $\tan\beta = \frac{3}{5}$

$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$

$= \frac{\frac{1}{4} + \frac{3}{5}}{1 - \frac{3}{20}}$ ✓

$= \frac{5+12}{20-3}$

$= 1$ ✓

(d)

$3 + x(2x-1)(x+1) \equiv 2x^3 + bx^2 + cx + 3$

$+x(2x^2 + x - 1) \equiv 2x^3 + bx^2 + cx + 3$

$2x^3 + x^2 - x + 3 \equiv 2x^3 + bx^2 + cx + 3$ ✓

equating co-efficients:

$b = 1$ and $c = -1$. ✓

(e) (i) $y = e^{-x^2}$

$\frac{dy}{dx} = -2x\,e^{-x^2}$ ✓

(ii) $\frac{d^2y}{dx^2} = -2e^{-x^2} + -2xe^{-x^2} \times -2x$

$= -2e^{-x^2} + 4x^2e^{-x^2}$ ✓

$= 2e^{-x^2}(2x^2 - 1)$

(iii) find the x-coordinates of the points of inflexion.

$2e^{-x^2}(2x^2-1) = 0$ ✓

$2x^2 = 1$

$x^2 = \frac{1}{2}$

$x = \pm\frac{1}{\sqrt{2}}$

The curve is concave down $\frac{-1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$ ✓

(from the graph.)

OR.

Solve $2e^{-x^2}(2x^2-1) < 0$

$2x^2 - 1 < 0$

$x^2 < \frac{1}{2}$

$\frac{-1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$

(12)

QUESTION 2

(a) $x = 5 + 4t - t^2$

(i) when $t = 0$, $x = 5$ ✓

(ii) $x = 5 + 4t - t^2$

$\dot{x} = 4 - 2t$

$4 - 2t = 0$

$2t = 4$

$t = 2s$. ✓

The particle changes direction at $x = 5 + 8 - 4$

$x = 9$. ✓

(iii) when $t = 0$, $x = 5$

when $t = 2$, $x = 9$

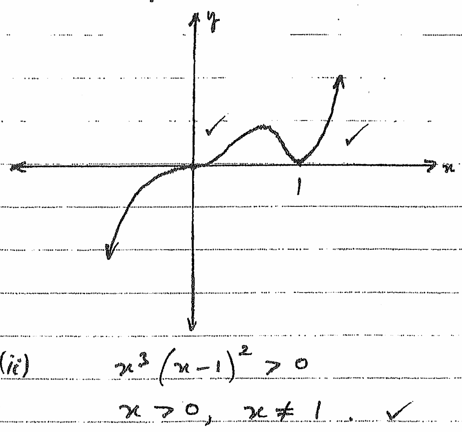
when $t = 6$, $x = -7$

distance $= 5 + 9 + 7$

travelled $= 20m$. ✓

(b) (i) $y = x^3(x-1)^2$

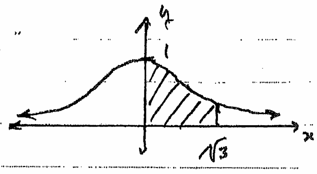
x intercepts at $x = 0$ and $x = 1$



(ii) $x^3(x-1)^2 > 0$

$x > 0$, $x \neq 1$. ✓

(c)



$$A = \int_0^{\sqrt{3}} \frac{dx}{1 + x^2} ✓$$

$$= \left[\tan^{-1} x\right]_0^{\sqrt{3}}$$

$$= \tan^{-1}\sqrt{3} - \tan^{-1} 0$$

$$= \frac{\pi}{3} \text{ square units.} ✓$$

(d)

(i) $\int \sec^2 3x \, dx = \frac{1}{3}\tan 3x + c$ ✓

(ii) $\int \sin^3 x \cos x \, dx = \frac{1}{4}\sin^4 x + c$ ✓✓

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QUESTION 3

(a) $\cos^2 x - \sin^2 x = \frac{\sqrt{3}}{2}$

for $0 \le x \le 2\pi$.

$\cos 2x = \frac{\sqrt{3}}{2}$ ✓

✓ $2x = \frac{\pi}{6}, \frac{11\pi}{6}, \frac{13\pi}{6}, \frac{23\pi}{6}$

✓ $x = \frac{\pi}{12}, \frac{11\pi}{12}, \frac{13\pi}{12}, \frac{23\pi}{12}$

(b) (i) $v = 2 - 4e^{-t}$

when $t = 0$, $v = 2 - 4$

$v = -2$ m/s. ✓

(ii) when $t = \ln 2$

$v = 2 - 4e^{-\ln 2}$

$v = 2 - 4e^{\ln \frac{1}{2}}$ ✓

$v = 2 - 4 \times \frac{1}{2}$

$v = 0$.

(iii) $\int_0^{\ln 2} 2 - 4e^{-t}\, dt$

$= \left[2t + 4e^{-t}\right]_0^{\ln 2}$ ✓

$= (2\ln 2 + 4e^{-\ln 2}) - (0 + 4)$

$= 2\ln 2 + 4 \times \frac{1}{2} - 4$ ✓

$= |\ln 4 - 2|$ m

(iv) as $t \to \infty$, $e^{-t} \to 0$

$v = 2 - 4e^{-t}$

$v \to 2$ m/s. ✓

(e)

(i) $f(x) = \ln \frac{(1 + \sin x)}{\cos x}$

$f(x) = \ln(1 + \sin x) - \ln \cos x$

$f'(x) = \frac{\cos x}{1 + \sin x} + \frac{\sin x}{\cos x}$ ✓

$= \frac{\cos^2 x + \sin x(1 + \sin x)}{\cos x(1 + \sin x)}$

$= \frac{\cos^2 x + \sin^2 x + \sin x}{\cos x(1 + \sin x)}$

$= \frac{1 + \sin x}{\cos x(1 + \sin x)}$ ✓

$= \frac{1}{\cos x}$

$f'(x) = \sec x$.

(ii) $\int_0^{\frac{\pi}{4}} \sec x\, dx = \left[\ln\left(\frac{1 + \sin x}{\cos x}\right)\right]_0^{\frac{\pi}{4}}$ ✓

$= \ln\left(\frac{1 + \frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}}\right) - \ln 1$ ✓

$= \ln(\sqrt{2} + 1)$

12

QUESTION 4

(a) $x^3 - 4x^2 + 3x - 1 = 0$

(i) $\alpha + \beta + \gamma = -\frac{b}{a}$

$= 4$ ✓

(ii) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma}$

$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$

$= 3$

$\alpha\beta\gamma = -\frac{d}{a}$

$= 1$

$\therefore \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{3}{1}$

$= 3.$ ✓

(iii) $\alpha^2 + \beta^2 + \gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)$ ✓

$= 4^2 - 2 \times 3$

$= 10$ ✓

(b) (i) $x = 6\cos 2t$

$\dot{x} = -12\sin 2t$

Maximum velocity $|\dot{x}| = 12$ m/s ✓

(ii) $\ddot{x} = -24\cos 2t$

So $\ddot{x} = 0$ when

$-24\cos 2t = 0$ ✓

at $2t = \frac{\pi}{2}$

$t = \frac{\pi}{4}$ s. ✓

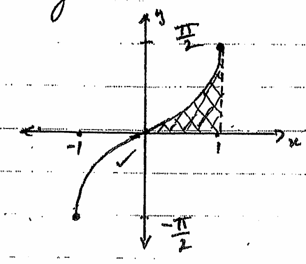
(iii) $\ddot{x} = -24\cos 2t$

$= -4(6\cos 2t)$

$= -4x.$ ✓

(c) $y = \sin^{-1} x$

(i)



✓✓

(ii) $y = \sin^{-1} x$

$x = \sin y$, $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

Area shaded = Rectangle − area bounded by the curve and the y-axis.

$A = 1 \times \frac{\pi}{2} - \int_0^{\frac{\pi}{2}} \sin y \, dy$ ✓

$= \frac{\pi}{2} + \left[\cos y\right]_0^{\frac{\pi}{2}}$

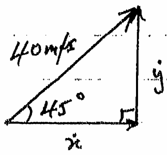
$= \frac{\pi}{2} + (0 - 1)$ ✓

$= \frac{\pi}{2} - 1$ square units.

12

QUESTION 5

(i) initial components of velocity:



$\dot{x} = 20\cos 45°$ $\qquad$ $\dot{y} = 40\sin 45°$

$\dot{x} = 20\sqrt{2}$ m/s $\qquad$ $\dot{y} = 20\sqrt{2}$ m/s. ✓

$\ddot{x} = 0$ $\qquad$ $\ddot{y} = -10$

$\dot{x} = c_1$ $\qquad$ $\dot{y} = -10t + c_2$

when $t = 0$, $\dot{x} = 20\sqrt{2}$ and $\dot{y} = 20\sqrt{2}$.

$c_1 = c_2 = 20\sqrt{2}$

$\dot{x} = 20\sqrt{2}$ $\qquad$ $\dot{y} = -10t + 20\sqrt{2}$

$x = \int 20\sqrt{2}\, dt$ $\qquad$ $y = \int -10t + 20\sqrt{2}\, dt$

$x = 20t\sqrt{2} + c_3$ ✓ $\qquad$ $y = -5t^2 + 20t\sqrt{2} + c_4$

when $t = 0$, $x = 0$ and $y = 22\frac{1}{2}$.

$c_3 = 0$ $\qquad$ $c_4 = 22\frac{1}{2}$

$x = 20t\sqrt{2}$ ✓ $\qquad$ $y = -5t^2 + 20t\sqrt{2} + 22\frac{1}{2}$

are the equations of motion for the shell.

(ii) Put $x = 180$ m.

$20t\sqrt{2} = 180$

$t = \frac{9\sqrt{2}}{2}$ s ✓

when $x = 180$.

$y = -5t^2 + 20t\sqrt{2} + 22\frac{1}{2}$

$= -5 \times \frac{81 \times 2}{4} + 20 \times \frac{9\sqrt{2}}{2} \times \sqrt{2} + 22\frac{1}{2}$

$= -\frac{401}{2} + 180 + 22\frac{1}{2}$

$= 202\frac{1}{2} + 202\frac{1}{2}$

$= 0$ ✓

When $x = 180$, $y = 0$. The shell hits the ground 180m from the base of the cliff.

(iii) The maximum height is reached when $\dot{y} = 0$.

$-10t + 20\sqrt{2} = 0$

$t = 2\sqrt{2}$ s. ✓

$y = -5t^2 + 20t\sqrt{2} + 22\frac{1}{2}$

$y = -5 \times 8 + 80 + 22\frac{1}{2}$

$y = 62\frac{1}{2}$ m ✓

The maximum height above the ground reached by the shell is $62\frac{1}{2}$ m.

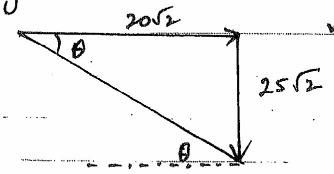
(iv) The shell strikes the ground at $t = \frac{9\sqrt{2}}{2}$ s. (part (i))

Components of velocity:

$\dot{x} = 20\sqrt{2}$

$\dot{y} = -10 \times \frac{9\sqrt{2}}{2} + 20\sqrt{2}$

$\dot{y} = -25\sqrt{2}$ ✓



$\tan\theta = \frac{25\sqrt{2}}{20\sqrt{2}}$

$\tan\theta = \frac{5}{4}$ ✓

(b) The equations of motion are:

$x = 40t\cos\alpha$ and $y = 40t\sin\alpha - 5t^2 + 22\frac{1}{2}$

put $x = 160$ and $y = 0$:

$40t\cos\alpha = 160$

$t = \frac{4}{\cos\alpha}$ substitute. ✓

$40t\sin\alpha - 5t^2 + 22\frac{1}{2} = 0$

$160\tan\alpha - \frac{80}{\cos^2\alpha} + 22.5 = 0$

$160\tan\alpha - 80\sec^2\alpha + 22.5 = 0$

$160\tan\alpha - (1 + \tan^2\alpha)80 + 22.5 = 0$

$-80\tan^2\alpha + 160\tan\alpha - 80 + 22.5 = 0$

$80\tan^2\alpha - 160\tan\alpha + 57.5 = 0$ ✓

using the quadratic formula

$\tan\alpha = \frac{160 \pm \sqrt{160^2 - 4 \times 80 \times 57.5}}{160}$

$\tan\alpha = \frac{160 \pm \sqrt{7200}}{160}$

$\tan\alpha = \frac{160 - 60\sqrt{2}}{160}$

$\alpha = 25°9'$ ✓

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QUESTION 6

(a) $v^2 = 6 + 10x - 4x^2$

(i) $\frac{1}{2}v^2 = 3 + 5x - 2x^2$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 5 - 4x$ ✓

$\therefore \ddot{x} = -4\left(x - \frac{5}{4}\right)$

$\ddot{x} \propto x$, the motion is SHM.

(ii) (α) centre of motion is $x = \frac{5}{4}$. ✓

(β) $n = 2$. $T = \frac{2\pi}{n}$

$T = \pi$ s ✓

(γ) when $v = 0$.

$6 + 10x - 4x^2 = 0$

$2x^2 - 5x - 6 = 0$

$(2x + 1)(x - 3) = 0$

$x = -\frac{1}{2}$ or $x = 3$ ✓

$-\frac{1}{2}$ 0 1 $\frac{5}{4}$ 2 3

amplitude $= 3 - \frac{5}{4}$

$= \frac{7}{4}$ ✓

(b) $\int_0^K \frac{6}{\sqrt{25 - 9x^2}}\, dx = \frac{\pi}{3}$

$6\left[\frac{1}{3}\sin^{-1}\frac{3x}{5}\right]_0^K = \frac{\pi}{3}$ ✓✓

$\left[\sin^{-1}\frac{3x}{5}\right]_0^K = \frac{\pi}{6}$

$\sin^{-1}\frac{3K}{5} = \frac{\pi}{6}$

$\frac{3K}{5} = \frac{1}{2}$

$\therefore 6K = 5$

$K = \frac{5}{6}$. ✓

(c) $t = \tan\frac{\theta}{2}$

$\cos\theta = \frac{1 - t^2}{1 + t^2}$

$\cos\theta + \tan\frac{\theta}{2} - 1 = 0$

$\frac{1 - t^2}{1 + t^2} + t - 1 = 0$ ✓

$1 - t^2 + t(1 + t^2) - (1 + t^2) = 0$

$1 - t^2 + t + t^3 - 1 - t^2 = 0$

$t^3 - 2t^2 + t = 0$

$t(t^2 - 2t + 1) = 0$

$t(t - 1)(t - 1) = 0$

$t = 0$ or $t = 1$ ✓

$\tan\frac{\theta}{2} = 0$ or $\tan\frac{\theta}{2} = 1$

$\frac{\theta}{2} = n\pi$ ✓ $\frac{\theta}{2} = \frac{\pi}{4} + \pi n$ ✓

$\theta = 2\pi n$ or $\theta = \frac{\pi}{2} + 2\pi n$

(where n is an integer)

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QUESTION 7.

(a) Using the division algorithm:

$P(x) = (x^2+x-2)Q(x) + ax + b$

$P(x) = (x-1)(x-2)Q(x) + ax + b$

$P(1) = 3$ and $P(-2) = -2$ ✓

① $a + b = 3$

② $-2a + b = -2$

①−② $3a = 5$

$a = \frac{5}{3}$ and $b = \frac{4}{3}$, hence the remainder is $\frac{5}{3}x + \frac{4}{3}$. ✓

(b) Let the tangent be $y = mx + b$. Since it passes through the point $(2,-8)$.

$2m + b = -8$

$b = -2m - 8$. ✓

$$\begin{cases} y = mx - 8 - 2m & ① \\ y = x^3 - 4x^2 - x + 2. & ② \end{cases}$$

Solving simultaneously:

$x^3 - 4x^2 - (1+m)x + (10+2m) = 0$. ✓

The roots of this cubic α, α and 2 correspond to the x co-ordinates of the points of intersection.

Sum of roots: $\alpha + \alpha + 2 = 4$

$2\alpha = 2$

$\alpha = 1$.

Therefore $Q\,(1,-2)$. ✓

(c) $\sqrt{\frac{1+x}{1-x}} \times \sqrt{\frac{1+x}{1+x}} = \frac{1+x}{\sqrt{1-x^2}}$ ✓

$\int \sqrt{\frac{1+x}{1-x}}\,dx = \int \frac{1+x}{\sqrt{1-x^2}}\,dx$

$= \int \frac{1}{\sqrt{1-x^2}} + \frac{x}{\sqrt{1-x^2}}\,dx$ ✓

$= \sin^{-1}x - \sqrt{1-x^2} + c$ ✓

(d)(i) $f(x) = e^x$

at $x=0$, $f'(x)$ e^x, $f'(0) = 1$

Using the definition. ✓

$$f'(x) = \lim_{h \to 0} \frac{e^{x+h} - e^x}{h}$$

$$f'(0) = \lim_{h \to 0} \frac{e^h - 1}{h}.$$

✓

Therefore $\lim_{h \to 0} \frac{e^h - 1}{h} = 1 \ldots (1$

(ii) $\lim_{n \to \infty} \frac{e^{1/n} + e^{2/n} + e^{3/n} + \cdots + e^{n/n}}{n}$

The numerator is a GP.

$S_n = \frac{a(r^n - 1)}{r-1}$

$= e^{1/n}\left\{ \frac{(e^{1/n})^n - 1}{e^{1/n} - 1} \right\}$

$= (e-1)\frac{e^{1/n}}{e^{1/n} - 1}$

$$\lim_{n \to \infty} \frac{e^{1/n} + e^{2/n} + e^{3/n} + \cdots + e^{n/n}}{n}$$

$$= \lim_{n \to \infty} \frac{(e-1)\,e^{1/n}}{n(e^{1/n} - 1)}$$ ✓

(Now let $\frac{1}{n} = h$ as $n \to \infty$, $h \to 0$)

$$= \lim_{h \to 0} \frac{(e-1)\, e^{h}}{e^{h}-1} \cdot h$$

$$= \lim_{h \to 0} (e-1)\, e^{h} \cdot \frac{h}{e^{h}-1}$$

$$= (e-1) \lim_{h \to 0} \frac{h}{e^{h}-1} \cdot e^{h}$$

from (1).

✓

$= e-1.$

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